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REFINED MEMBRANE THEORY OF ELECTROELASTIC SHELLS*

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An analysis of the membrane electroelastic state and the determination of the first vibration eigenfrequencies are often of particular interest in the analysis of thin-walled elements. It is shown how the error of membrane theory can be reduced considerably by introducing certain additional terms into the membrane boundary conditions.

1. To be specific, we will examine piezoceramic shells with thickness polarization. We will write the equations of the theory of the bending of piezoelectric shells to an accuracy of quantities of the order of $(\eta^1 + \eta^{2-2t})$, where t is the index of variability of the fundamental electroelastic state, and η is a small parameter equal to the ratio of half the shell thickness h and its characteristic dimension R:

The equations of equilibrium:

$$\frac{1}{A_i}\frac{\partial T_i}{\partial \alpha_i} + \frac{1}{A_j}\frac{\partial S}{\partial \alpha_j} + k_j(T_i - T_j) + 2k_iS - \frac{p}{R_i}N_i + 2h\rho\omega^2 u_i + X_i = 0 \quad (1.1)$$

$$\sum_{i=1}^{2}\left(\frac{T_i}{R_i} + p\frac{1}{A_i}\frac{\partial N_i}{\partial \alpha_i} + pk_jN_i\right) + 2h\rho\omega^2 w + Z = 0$$

$$N_i = \frac{1}{A_i}\frac{\partial G_i}{\partial \alpha_i} - \frac{1}{A_j}\frac{\partial H_{ij}}{\partial \alpha_j} + k_j(G_i - G_j) - k_i(H_{ij} + H_{ji}) \quad (1.2)$$

$$(k_i = (A_iA_j)^{-1}\,\partial A_i/\partial \alpha_j)$$

(the quantity p in (1.1) should be assumed equal to one; it is required later);

the electroelasticity relations:

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$$T_i = \frac{2Eh}{1-\nu^2}(\varepsilon_i + \nu_0\varepsilon_j) - \frac{2Fh}{1-\nu^2}E_3^{(0)}, \quad S = \frac{h}{(1+\nu)s_{11}^E}\omega \tag{1.3}$$

$$G_i = -\frac{2Bh^3}{3(1-\nu^2)}(\varkappa_i + \sigma\varkappa_j), \quad H_{ij} = \frac{2h^3}{3s_{11}^E(1+\nu)}\left(\tau - \frac{\omega}{2R_j}\right) \tag{1.4}$$

$$\nu = -\frac{s_{12}^E}{s_{11}^E}, \quad B = \frac{2k_{31}^2(1-\nu)}{(1-\nu-2k_{31}^2)s_{11}^E}, \quad \sigma = \frac{\nu + k_{31}^2}{1-k_{31}^2}, \quad k_{31}^2 = \frac{d_{31}^2}{s_{11}^E\varepsilon_{33}^T}$$

the geometric relations:

$$\varepsilon_i = \frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i} + k_i u_j - \frac{w}{R_i}, \quad \omega = \sum_{i=1}^{2}\left(\frac{1}{A_i}\frac{\partial u_j}{\partial \alpha_i} - k_i u_i\right) \tag{1.5}$$

$$\varkappa_i = -\frac{1}{A_i}\frac{\partial \gamma_i}{\partial \alpha_i} - k_i\gamma_j, \quad \gamma_i = -\frac{1}{A_i}\frac{\partial w}{\partial \alpha_i} - \frac{u_i}{R_i} \tag{1.6}$$

$$\tau = -\frac{1}{A_i}\frac{\partial \gamma_j}{\partial \alpha_i} + k_i\gamma_i + \frac{1}{R_i}\left(\frac{1}{A_j}\frac{\partial u_i}{\partial \alpha_j} - k_j u_j\right), \quad i \neq j = 1, 2$$

In (1.1)-(1.6) u_i and w are the displacements of the middle-surface points along the coordinate lines α_i and the normal to the middle surface, respectively, $\varepsilon_i, \omega, \varkappa_i, \tau$ are strain components, T_i and S are forces, G_i and H_{ij} are the bending and twisting moments, N_i are transverse forces, and E_3 is the normal component of the electric field vector. The customary notation /1, 2/ is used for the physical constants. The following quantities are an exception: E, ν_0, F, whose formulas depend on the kind of electrical conditions on the shell faces. For shells with electrodes on the faces on which the value of the electric potential difference $2V$ is given, they are described by (1.7), and for shells without electrodes by (1.8)

$$E = 1/s_{11}^E, \quad \nu_0 = \nu, \quad F = d_{31}/s_{11}^E, \quad E_3^{(0)} = -V/h \tag{1.7}$$

$$E = B, \quad \nu_0 = \sigma, \quad F = 0, \quad E_3^{(0)} = 0 \tag{1.8}$$

We shall henceforth assume that the load acting on the shell, the shell geometry, and the conditions of supporting its edges are such that the conditions for partitioning the electroelastic state into the membrane state and into simple edge effects are satisfied.

2. We will write the asymptotic representation of the desired membrane electroelastic state parameters /1/

$$u_i/R = \eta^t u_{i*}, \quad w/R = w_*, \quad (T_i, S)/(2Eh) = T_{i*} \tag{2.1}$$
$$(\varepsilon_i, \omega) = (\varepsilon_{i*}, \omega_*), \quad (R\varkappa_i, R\tau) = \eta^{-2t}(\varkappa_{i*}, \tau_*)$$
$$(G_i, H_{ij})/2BhR = \eta^{2-2t}(G_{i*}, H_{ij*}), \quad N_i/2Bh = \eta^{2-3t}N_{i*}$$

The powers of the small parameter η are selected in such a manner that all the dimensionless desired quantities with the asterisks are of the same order.

We will substitute the asymptotic form (2.1) into (1.1)-(1.6), and in addition, stretch the scale along the coordinate lines α_i as is usual for asymptotic methods, in such a manner that differentiation with respect to the newly introduced variables ξ_i does not result in a substantial increase or decrease in the desired quantities

$$\alpha_i = \eta^t R\xi_i \tag{2.2}$$

We will neglect small terms to an accuracy of terms of order of magnitude ε_1, in the transformed Eqs.(1.1)-(1.6), where

$$\varepsilon_1 = O(\eta^1 + \eta^{2-4t}) \tag{2.3}$$

whereupon we obtain a system of membrane theory Eqs.(1.1), (1.3), (1.5) in which p should be set equal to zero.

3. The asymptotic form of the simple edge effect of electroelastic shells is analogous to the corresponding asymptotic form in the theory of non-elastic shells. We will write it for the edge effect of shells with thickness polarization at the edge $\alpha_i = \alpha_{i0}$:

$$u_i/R = \eta^{1/2}u_{i*}, \; u_j/R = \eta^{1-t}u_{j*}, \quad w/R = w_* \tag{3.1}$$
$$(G_1, G_2)/(2BhR) = \eta^1(G_{1*}, G_{2*}), \quad H_{ij}/(2BhR) = \eta^{3/2-t}H_{ij*}$$
$$N_i/(2Eh) = \eta^1 N_{i*}, \quad N_j/(2Eh) = \eta^{1-t}N_{j*},$$

$$T_j/(2Eh) = \eta^0 T_{j*}, \quad T_i/(2Eh) = \eta^{1/2} T_{i*}, \quad S/(2Eh) = \eta^{1/2-t} S_*$$

With respect to the variability of the desired quantities, it is assumed, exactly as in the theory of non-electric shells, that the variablity of the desired quantities in the direction orthogonal to the edge exceeds the variability along the edge

$$\alpha_i = \eta^{1/2} R \xi_i, \quad \alpha_j = \eta^t R \xi_j \tag{3.2}$$

Constructing the equations of the simple edge effect just as is done in the theory of non-electric shells, we obtain the following simple edge effect equations in shells with thickness polarization

$$\frac{1}{A_i^4} \frac{\partial^4 w}{\partial \alpha_i^4} + 4 g_i^4 w = 0, \quad u_i = -\left(\frac{1}{R_i} + \frac{\nu_0}{R_j}\right) \frac{1}{4 g_i^4} \frac{1}{A_i^3} \frac{\partial^3 w}{\partial \alpha_i^3} \tag{3.3}$$

$$\frac{1}{A_i} \frac{\partial}{\partial \alpha_i} \left[\frac{1}{A_i} \frac{\partial u_j}{\partial \alpha_i} + \frac{1}{A_j} \frac{\partial u_i}{\partial \alpha_j} - k_i u_i\right] = 2(1+\nu) s_{11}^E E \left(\frac{1}{A_j} \frac{\partial}{\partial \alpha_j} - k_i\right) \frac{w}{R_j}$$

$$G_i = -\frac{2Bh^3}{1-\nu^2} \frac{1}{A_i^2} \frac{\partial^2 w}{\partial \alpha_i^2}, \quad G_j = \sigma G_i, \quad N_i = \frac{1}{A_i} \frac{\partial G_i}{\partial \alpha_i}$$

$$T_j = -2Eh \frac{w}{R_j}, \quad T_i = -k_i R_j N_i, \quad S = \frac{h}{(1+\nu) s_{11}^E} \omega, \quad H_{ij} = \frac{2h^3}{3 s_{11}^E (1+\nu)} \tau,$$

$$\omega = \frac{1}{A_1} \frac{\partial u_2}{\partial \alpha_1} + \frac{1}{A_2} \frac{\partial u_1}{\partial \alpha_2} - k_i u_i, \quad \tau = -k_i \frac{1}{A_i} \frac{\partial w}{\partial \alpha_i} + \frac{1}{A_i} \frac{\partial}{\partial \alpha_i} \frac{1}{A_j} \frac{\partial w}{\partial \alpha_j}$$

$$4 g_i^4 = \frac{3(1-\sigma^2) E}{h^2 R_j^2 B} (1-\lambda), \quad \lambda = \frac{\rho \omega^2 R_j^2}{E}$$

where ν_0 and E are given by (1.7) and (1.8), depending on the electrical conditions on the faces. Formulas (3.3) are suitable for both statics and dynamics. If $g_i^4 > 0$ in dynamic problems, then (3.3) describe the dynamic edge effect; if $g_i^4 < 0$ then (3.3) describe a rapidly varying oscillating electroelastic state.

Formulas (3.3) enable us to determine the desired quantities to the accuracy of quantities of order ε_2, where

$$\varepsilon_2 = O(\eta^{1/2-t}) \tag{3.4}$$

4. The error in the membrane theory equations is determined by (2.3). To estimate the error of the membrane boundary conditions and to refine them, we follow /3/ and represent each of the desired quantities P in the form of the sum

$$P = P^{(f)} + \eta^a P^{(e)} \tag{4.1}$$

The superscripts (f) and (e) denote that the given quantity belongs to a membrane electroelastic state or to a simple edge effect, respectively.

We shall consider the shell loaded by a surface electrical and mechanical load that is taken into account when solving the membrane problem. Consequently, an inhomogeneous system of equations holds for determining $P^{(f)}$. The quantity $P^{(e)}$ is found from the homogeneous edge effect equations, and consequently, there is a scale factor η^a, for $P^{(e)}$ where a is a number that is common for all the quantities that will be chosen depending on the boundary conditions.

Let us examine the boundary conditions on a rigidly clamped edge $\alpha_i = \alpha_{i0}$. Taking account of (2.1), (3.1), (4.1), they can be written in the form

$$u_{i*}^{(f)} + \eta^{a+1/2} u_{i*}^{(e)} = 0, \quad u_{j*}^{(f)} + \eta^{a+1-t} u_{j*}^{(e)} = 0 \tag{4.2}$$

$$w_*^{(f)} + \eta^a w_*^{(e)} = 0, \quad \gamma_{i*}^{(f)} + \eta^{a+1/2} \gamma_{i*}^{(e)} = 0$$

The boundary-value problem obtained can be solved by setting $a = 0$. Retaining the principal terms in (4.2), we obtain the traditional tangential boundary conditions for membrane theory

$$u_{i*}^{(f)} = 0, \quad u_{j*}^{(f)} = 0 \tag{4.3}$$

and for the simple edge effect

$$w_*^{(f)} = -w_*^{(e)}, \quad \gamma_{i*}^{(e)} = 0 \tag{4.4}$$

An error $O(\eta^{1/2})$, greater than the error of the membrane Eqs.(2.3) was committed here in the boundary conditions (4.3), (4.4), when neglecting small terms.

To refine the boundary conditions we rewrite (4.2) in the form

$$u_{i*}^{(f)} + \eta^{1/2} u_{i*}^{(e)} = 0, \quad u_{j*}^{(f)} = 0 \tag{4.5}$$

The solution of the resolving Eq.(3.3) at the edge $\alpha_i = \alpha_{i0}$ $(\alpha_i \leqslant \alpha_{i0})$ has the form

$$w^{(e)} = [f_1 \cos A_i g_i (\alpha_i - \alpha_{i0}) + f_2 \sin A_i g_i (\alpha_i - \alpha_{i0})]\, e^{A_i g_i(\alpha_i - \alpha_{i0})}$$

By satisfying conditions (4.4) we find on the edge $\alpha_i = \alpha_{i0}$

$$f_1 = -f_2 = -w^{(f)}, \quad u_i^{(e)} = -\left(\frac{1}{R_i} + \frac{\nu_0}{R_j}\right)\frac{w^{(f)}}{2g_i}$$

By substituting the value of $u_i^{(e)}$ obtained in the first formula into (4.5), we obtain the boundary conditions of membrane theory for a rigidly clamped edge, refined to quantities $O(\eta^{1-t})$

$$u_i^{(f)} - \frac{1}{2g_i}\left(\frac{1}{R_i} + \frac{\nu_0}{R_j}\right) w^{(f)} = 0, \quad u_j^{(f)} = 0$$

The membrane boundary conditions on a hinge-supported edge can be refined in an analogous manner

$$T_i^{(f)} + \frac{hk_jE}{R_jg_i} w^{(f)} = 0, \quad u_j^{(f)} = 0$$

As a result of partitioning the electroelastic state for shells with preliminary tangential polarization /4/, it has been shown that the membrane problem is a coupled electroelastic problem described by a system of sixth-order equations. We present without derivation the refined boundary conditions on the clamped edge $\alpha_i = \alpha_{i0}$, by considering the shell to be polarized initially along the α_2-line of the middle surface

$$u_i^{(f)} - \left(\frac{1}{R_i} + \frac{\nu_i + b_i}{R_j}\right)\frac{1}{g_i} w^{(f)} = 0, \quad u_j^{(f)} = 0 \tag{4.6}$$

$$\psi^{(f)}_{\alpha_i = \alpha_{i0}} = V \tag{4.7}$$

$$D_1^{(f)} - \frac{2h^2}{3} n_{11} d_{33}^T \frac{1}{A_2} \frac{\partial}{\partial \alpha_2}[R_2 g_1^3 w^{(f)}] = 0, \quad (\alpha_1 = \alpha_{10}) \tag{4.8}$$

$$D_2^{(f)} + d_{33} n_{11} k_1 a_2 g_2^{-1} w^{(f)} = 0 \quad (\alpha_2 = \alpha_{20}) \tag{4.9}$$

Here

$$4g_i^4 = \frac{3}{h^2 n_{ii}}\left(\frac{n_{jj} a_i}{R_j^2} - \rho w^2\right)$$

$$n_{11} = s_{33}^E/\delta, \quad n_{22} = s_{11}^E/\delta, \; n_{ij} = s_{13}^E/\delta, \; \delta = s_{11}^E s_{33}^E - (s_{13}^E)^2$$

$$\nu_i = n_{12}/n_{ii}, \quad b_1 = 0, \quad b_2 = c_2 d_{31} n_{11} a_2/(n_{22}\varepsilon_{33}^T)$$

$$a_1 = 1 - \nu_1\nu_2, \quad a_2 = a_1\varepsilon_{33}^T/[\varepsilon_{33}^T + d_{31}(\nu_2 c_2 - c_1)]$$

$$c_1 = (d_{31}s_{33}^E - d_{33}s_{13}^E)/\delta, \quad c_2 = (d_{33}s_{11}^E - d_{31}s_{13}^E)/\delta$$

Condition (4.7) should be satisfied on an edge with electrodes, while the electrical conditions (4.7) and (4.8) should be satisfied on the edges $\alpha_1 = \alpha_{10}$ and $\alpha_2 = \alpha_{20}$, respectively.

The first of conditions (4.6) in the refined membrane conditions (4.6)-(4.9) should be replaced by the condition for the force T_i

$$T_i + hn_{jj}k_j a_i R_j^{-1} w^{(f)} = 0$$

on the hinge-supported edge of a shell with tangential polarization.

The refined membrane boundary conditions on an edge without clamping will be identical with the usual tangential boundary conditions.

Thus, if the membrane equations are supplemented by refined boundary conditions, the error will be reduced from a magnitude $O(\eta^{1/2})$ to a magnitude $O(\eta^{1-t})$.

A simple edge effect can be constructed to the same accuracy by using an iteration process that reduces to elementary integrable equations just as is done in the theory of non-electrical shells. To achieve the accuracy of the refined membrane theory here it is sufficient to combine ourselves to the first two approximations.

Expanding each of the desired quantities in a series of the form

$$P_* = \sum_{n=0}^{\infty} k^{-n} P^{(n)}, \quad k = \eta^{-1/2}$$

where P_* should be understood to be any of the quantities $u_{i*}, \ldots, S_*$ introduced by (3.1), substituting the expansions into the system of shell theory equations, making the substitution (3.2) and equating coefficients of identical powers of k, we obtain equations for different approximations.

We will write the fundamental equations of the two first approximations for shells with thickness polarization with faces with electrodes at the edge $\alpha_i = \alpha_{i0}$:

$$\frac{1}{a_{1,0}^4}\frac{\partial^4 w^{(n)}}{\partial \xi_i^4} + \frac{3R^2(1-\lambda)(1-\nu^2)}{r_{j,0}^2 s_{11}^E B} w^{(n)} = \xi_i \frac{6R^2(1-\nu^2)}{r_{j,0} s_{11}^E B} \times$$

$$\left(\frac{2a_{\ ,0}(1-\lambda)}{a_{i,1}} - \frac{r_{j,0}}{r_{j,1}}\right) w^{(n-1)} - 2\left(\frac{3}{a_{i,1}} + k_{j,0}R\right)\frac{1}{a_{i,0}^3}\frac{\partial^3 w^{(n-1)}}{\partial \xi_i^3}$$

$$\frac{1}{a_{i,0}}\frac{1}{R}\frac{\partial u_i^{(n)}}{\partial \xi_i} - \left(\frac{1}{r_{i,0}} + \frac{\nu}{r_{j,0}}\right) w^{(n)} = -\frac{\xi_i}{a_{i,1}}\frac{1}{R}\frac{\partial u_i^{(n-1)}}{\partial \xi_i} +$$

$$\xi_i\left(\frac{1}{r_{i,1}} + \frac{\nu}{r_{j,1}}\right) w^{(n-1)} - \nu k_{j,0} u_i^{(n-1)} +$$

$$\frac{k_{j,0} r_{j,0}(1-\nu^2) s_{11}^E B}{3R(1-\nu^2)(1-\lambda)}\frac{1}{a_{i,0}^3}\frac{\partial^3 w^{(n-1)}}{\partial \xi_i^3} \quad (n = 0, 1)$$

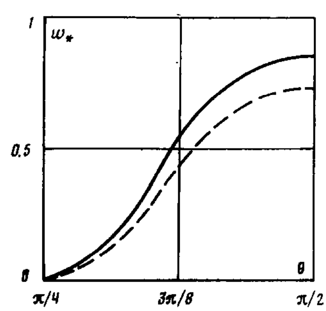
The superscript in parentheses denotes the number of the approximation, while quantities with negative superscripts should be considered to be zero.

The quantities characterizing the metric and geometry of the shell middle surface $1/A_i$, $1/R_i$, k_i are expanded in a Taylor series in powers of ξ_i in the form

$$Q = \sum_{n=0}^{\infty}\left(\frac{\xi}{k}\right)^n Q_n$$

Here Q is any of the quantities $1/A_i$, $1/R_i$, k_i and Q_n are coefficients of the Taylor series, denoted, respectively, by $1/a_{i,n}$, $1/r_{i,n}$ and $k_{i,n}$.

5. As an illustration we analyse a part of a spherical shell with two rigidly clamped edges that coincide with the parallel $\theta_1 = \pi/4$ and $\theta_2 = 3\pi/4$ in the geographical coordinate system. The shell is made from the piezoceramic PZT-4 with thickness polarization and with an electrical potential $2V$ given on its faces with electrodes. The results of calculating the deflection $w_* = wh/(rd_{31}V)$ are presented in the figure where the dashed line denotes the deflection calculated by a membrane theory of the type in /3/ and the theory of the first approximation of a simple edge effect, while the solid line shows the deflection calculated by the refined membrane theory taking two approximations of the simple edge effect into account. In addition, a computation of the deflection was preformed by the bending theory of shells by a numerical method and the results of the latter agree well with the solid line in the figure.



The asymptotic analysis and calculations performed have shown that the refined membrane theory ensures the same computation accuracy for electroelastic states with small index of variability as does the bending theory of shells. Note that all the results obtained remain valid for non-electrical shells.

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